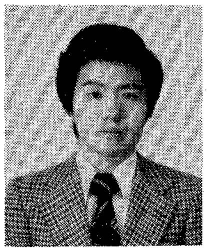

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Uniform Asymptotic Technique for Analyzing Wave Propagation in Inhomogeneous Slab Waveguides

HIROYOSHI IKUNO AND AKIRA YATA

Abstract—**The guided modes of inhomogeneous dielectric slab waveguides are analyzed by a uniform asymptotic technique based on the related equation method. This technique gives highly accurate solutions in the sense of asymptotic expansion. The algorithm for calculating the guided modes of slab waveguides with an even polynomial refractive-index medium is presented. As an example, we calculate the third-order approximate solutions for the guided modes in an analytic form. The results show that the WKB solutions for higher order modes are more accurate than for the lower order modes and the correction to the WKB solutions is significant for the lower order modes. The numerical result for eigenvalues and modal fields confirms that the third-order asymptotic solution is accurate for all the guided modes of the near-parabolic profile waveguides and for higher order modes in the case of the quasi-Gaussian profile.**

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I. Introduction

Recent advances of fabrication technology of optical integrated circuits produce optical channel waveguides and directional couplers with a great variety of inhomogeneous media including those with a Gaussian distribution. A number of design theories have been presented to evaluate the propagation characteristics of such inhomogeneous slab waveguides [1]–[4]. Although the WKB method is useful for analyzing these waveguides, it fails in the case of relatively strong inhomogeneity [4]. Considerable efforts have been made to overcome this drawback of the WKB method [5]–[9].

In this paper, we analyze the guided modes of waveguides with an even polynomial-profile medium. This pro-

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file describes a considerably general class of refractive-index distributions such as the Gaussian profile. We adopt a uniform asymptotic technique based on the related equation method [9] and derive a formula for calculating the Nth-order correction to the WKB solutions: the propagation constants are obtained by solving the refined form of the Bohr–Sommerfeld quantum condition, and the modal fields are expressed in terms of the parabolic cylinder functions [10]. As a practical example, we calculate the third-order asymptotic solution in an analytic form. The accuracy and the convergence of the approximate solution are examined numerically in the near-parabolic profile case and in the quasi-Gaussian profile case.

II. Formulation of Problem

We consider the two-dimensional waveguides, in which the refractive-index is represented as

$$n(x)=n_0(1-h(x))^{1/2}$$
$$h(x)=(gx)^2-a_2(gx)^4+a_3(gx)^6+\cdots+(-1)^{M+1}a_M(gx)^{2M} \quad (1)$$

where n_0 is the refractive-index at $x=0$, g is a positive constant, and a_M's are constants such that $h(x)$ increases monotonically. Here we use a Cartesian coordinate system (x, y, z). The guided waves propagate along the z-axis according to $\exp(j(\omega t-\beta_m z))$, where β_m are propagation constants and m is the mode index $(m=0,1,2,\cdots,)$. The fields of the TE modes and the TM modes can be described in terms of the y-components of the electric field E_y and the magnetic field H_y, respectively. Now we put E_y and H_y in the form

$$E_y(x,z,t)=\Phi_m(x)\exp(j(\omega t-\beta_m z)), \qquad \text{for TE-modes} \quad (2a)$$

and

$$H_y(x,z,t)=n(x)\Phi_m(x)\exp(j(\omega t-\beta_m z)), \qquad \text{for TM-modes}$$
$$\beta_m=k(1-b_m)^{1/2}, \qquad k=(2\pi/\lambda)n_0 \quad (2b)$$

where λ is the wavelength in vacuum. Then, the transverse mode functions $\Phi_m(x)$ satisfy the following equation:

$$\Phi_m''(x)+k^2Q(x)\Phi_m(x)=0$$
$$Q(x)=b_m-h(x)+\begin{cases}0, & \text{for TE-modes}\\ -(1/k^2)(1-h(x))^{1/2}\left((1\text{-}h(x))^{-1/2}\right)'', & \text{for TM-modes}\end{cases} \quad (3)$$

where the prime denotes the derivative with respect to x. The problem of determining the guided modes is to calculate the eigenvalues b_m and the modal fields $\Phi_m(x)$.

III. Algorithm of the Nth-Order Approximate Solution

Let us construct the formula for obtaining the Nth-order approximate solution of (3). It is intended that such a solution is in the refined form of the Bohr–Sommerfeld quantum condition for the eigenvalue and in the form of a parabolic cylinder function for the modal field. The construction method adopted here is based on the repeated use of the Langer transformation [9]. Setting $\phi_1=\Phi_m$ in (3), we have

$$\epsilon^2\frac{d^2}{dw_1^2}\phi_1(w_1)+Q_1(w_1)\phi_1(w_1)=0$$
$$Q_1(w_1)=b_m-h(w_1)+\begin{cases}0, & \text{for TE-modes}\\ -\epsilon^2(1-h(w_1))^{1/2}\dfrac{d^2}{dw_1^2}\left((1-h(w_1))^{-1/2}\right), & \text{for TM-modes}\end{cases} \quad (4)$$

where $w_1=gx$ and $\epsilon=g/k$. First, we transform w_p to w_{p+1} $(p=1,2,\cdots,N)$ through the relations

$$I_p\int_0^{w_{p+1}}\sqrt{1-w_{p+1}^2}\,dw_{p+1}=\int_0^{w_p}\sqrt{Q_p(w_p)}\,dw_p, \qquad \text{for } 0\leqslant w_p\leqslant\xi_p$$
$$I_p\int_1^{w_{p+1}}\sqrt{w_{p+1}^2-1}\,dw_{p+1}=\int_{\xi_p}^{w_p}\sqrt{-Q_p(w_p)}\,dw_p, \qquad \text{for } w_p>\xi_p \quad (5a)$$

where

$$I_p=(4/\pi)\int_0^{\xi_p}\sqrt{Q_p(w_p)}\,dw_p \quad (5b)$$

$$Q_p(w_p)=\begin{cases}Q_1(w_1), & \text{for } p=1\\ I_{p-1}^2(1-w_p^2)-\epsilon^2R_{p-1}(w_p), & \text{for } p=2,3,\cdots,N\end{cases} \quad (5c)$$

$$R_{p-1}(w_p)=\left(\frac{dw_{p-1}}{dw_p}\right)^{1/2}\frac{d^2}{dw_p^2}\left(\left(\frac{dw_{p-1}}{dw_p}\right)^{-1/2}\right). \quad (5d)$$

In (5), $Q_p(w_p)$ is continuously differentiable at $w_p=\xi_p$, where ξ_p is a simple zero of $Q_p(w_p)$. Transformation (5) maps the regions $0\leqslant w_p\leqslant\xi_p$ and $\xi_p<w_p$ into $0\leqslant w_{p+1}\leqslant 1$ and $1<w_{p+1}$, respectively. Next, we consider the following transformation:

$$\phi_{p+1}(w_{p+1})=(dw_p/dw_{p+1})^{-1/2}\phi_p(w_p) \qquad (p=1,2,\cdots,N). \quad (6)$$

The application of the transformation (6) together with (5) N times to (4) yields

$$\epsilon^2\frac{d^2}{dw_{N+1}^2}\phi_{N+1}(w_{N+1})+\left(I_N^2(1-w_{N+1}^2)-\epsilon^2R_N(w_{N+1})\right)\phi_{N+1}(w_{N+1})=0. \quad (7)$$

By neglecting R_N in (7) (see (12)), we have

$$\epsilon^2\frac{d^2}{dw_{N+1}^2}\phi_{N+1}(w_{N+1})+I_N^2(1-w_{N+1}^2)\phi_{N+1}(w_{N+1})=0. \quad (8)$$

Provided that the mode index m is related to I_N as

$$I_N=(2m+1)\epsilon \quad (9)$$

solution of (8) vanishing at $w_{N+1}=\pm\infty$ can be expressed in terms of the parabolic cylinder functions [10] as

$$\phi_{N+1}(w_{N+1})=D_m(\sqrt{4m+2}\,w_{N+1}). \quad (10)$$

Then, from (6) and (10), the mode function of (4) can be represented as

$$\phi_1 = \left(\prod_{p=1}^{N} \left(dw_{p+1}/dw_p \right) \right)^{-1/2} D_m\left(\sqrt{4m+2}\, w_{N+1}\right). \quad (11)$$

Equations (9) and (11) are the approximate solutions of (4). Now, let us estimate the errors of (9) and (11) in (4). Here we consider only even polynomial refractive-index profiles. Then we have the estimate of R_N (see the Appendix) such that

$$R_N = O(\epsilon^{2N-1}) \quad (12)$$

where $O(\cdot)$ means an O-symbol [11]. From (5), (7), (9), (11), (12), and (A1), we get the Nth-order approximate solution of (3): for eigenvalues we have

$$b_m = b_m^{(N)} + O(\epsilon^{2N}) \quad (13a)$$

where $b_m^{(N)}$ is the solution of

$$(1/\epsilon) \int_0^{\xi_N} \sqrt{Q_N(w_N)}\, dw_N = (m+1/2)(\pi/2) \quad (13b)$$

and for the modal fields we obtain

$$\Phi_m(w_1) = \Phi_m^{(N)}(w_1) + O(\epsilon^{2N-1}) \quad (14a)$$

where $\Phi_m^{(N)}(w_1)$ is expressed in the form

$$\Phi_m^{(N)}(w_1) = \left(dw_{N+1}(w_1)/dw_1 \right)^{-1/2} \cdot D_m\left(\sqrt{4m+2}\, w_{N+1}(w_1)\right). \quad (14b)$$

This is a uniform asymptotic technique for calculating the Nth-order approximate eigenvalues $b_m^{(N)}$ and modal fields $\Phi_m^{(N)}$ in the sense of asymptotic expansion. Using these results, we can evaluate the propagation characteristics of the guided modes in even polynomial refractive-index media as precisely as one wishes.

In an actual calculation, we obtain $b_m^{(N)}$ and $w_{N+1}(w_1)$ in (14b) in the power series of ϵ with the help of a family of relations (A1). The terms up to a_{2N-1} in (1) are exactly considered in the Nth-order approximate solution. It is noted that (13b) is the refined form of the Bohr–Sommerfeld quantum condition. The WKB solution of (3) is the first-order approximate solution in this formulation and is represented by $b_m^{(1)}$ and $\Phi_m^{(1)}$. Transformation (5) is recursive. So we can solve (13) and (14) by using an analytic algebraic computer code.

IV. Example

As an example, we calculate the third-order asymptotic solution. In this case, the terms up to a_5 are considered. The computer code is described by BASIC of HP 9845B. The results are as follows:

$$\begin{aligned}
b_m^{(3)} = {} & s\epsilon - \left(\frac{3}{8} + \frac{3}{8}\left(\frac{1}{s^2}\right) \right) a_2 (s\epsilon)^2 - \left[\frac{17}{64} a_2^2 - \frac{5}{16} a_3 + \left(\frac{67}{64} a_2^2 - \frac{25}{16} a_3 \right)\left(\frac{1}{s^2}\right) \right] (s\epsilon)^3 \\
& - \left[\frac{375}{1024} a_2^3 - \frac{165}{256} a_2 a_3 + \frac{35}{128} a_4 + \left(\frac{1707}{512} a_2^3 - \frac{885}{128} a_2 a_3 + \frac{245}{64} a_4 \right)\left(\frac{1}{s^2}\right) \right. \\
& \left. + \left(\frac{1539}{1024} a_2^3 - \frac{945}{256} a_2 a_3 + \frac{315}{128} a_4 \right)\left(\frac{1}{s^4}\right) \right] (s\epsilon)^4 \\
& - \left[\frac{10689}{16384} a_2^4 - \frac{3129}{2048} a_2^2 a_3 + \frac{393}{1024} a_3^2 + \frac{189}{256} a_2 a_4 - \frac{63}{256} a_5 \right. \\
& + \left(\frac{89165}{8192} a_2^4 - \frac{29555}{1024} a_2^2 a_3 + \frac{4145}{512} a_3^2 + \frac{2205}{128} a_2 a_4 - \frac{945}{128} a_5 \right)\left(\frac{1}{s^2}\right) \\
& \left. + \left(\frac{305141}{16384} a_2^4 - \frac{117281}{2048} a_2^2 a_3 + \frac{19277}{1024} a_3^2 + \frac{10521}{256} a_2 a_4 - \frac{5607}{256} a_5 \right)\left(\frac{1}{s^4}\right) \right] (s\epsilon)^5 \\
& + \begin{cases} O(\epsilon^6), & \text{for TE-modes} \\
\left(\frac{1}{s^2}\right)(s\epsilon)^2 + (2 - 3a_2)\left(\frac{1}{s^2}\right)(s\epsilon)^3 + \left(\frac{21}{8} - \frac{45}{8} a_2 - \frac{9}{4} a_2^2 + \frac{45}{8} a_3 \right)\left(\frac{1}{s^2} + \frac{1}{s^4} \right)(s\epsilon)^4 \\
\quad + \left[\left(\frac{25}{8} - \frac{261}{32} a_2 - \frac{29}{16} a_2^2 + \frac{35}{4} a_3 - \frac{255}{64} a_2^3 + \frac{345}{32} a_2 a_3 - \frac{35}{4} a_4 \right)\left(\frac{1}{s^2}\right) \right. \\
\quad \left. + \left(\frac{109}{8} - \frac{1239}{32} a_2 - \frac{91}{16} a_2^2 + \frac{175}{4} a_3 - \frac{1005}{64} a_2^3 + \frac{1455}{32} a_2 a_3 - \frac{175}{4} a_4 \right)\left(\frac{1}{s^4}\right) \right] (s\epsilon)^5 \\
\quad + O(\epsilon^6), & \text{for TM-modes}
\end{cases}
\end{aligned} \quad (15)$$

$$(s\epsilon)^{1/2} w_4 = d_0 w_1 + d_1 w_1^3 + d_2 w_1^5 + d_3 w_1^7 + d_4 w_1^9 \quad (16)$$

with

$$d_0=1-\frac{3}{16}a_2 s\epsilon-\left[\frac{77}{512}a_2^2-\frac{5}{32}a_3+\left(\frac{91}{256}a_2^2-\frac{5}{8}a_3\right)\left(\frac{1}{s^2}\right)\right](s\epsilon)^2$$

$$-\left[\frac{1731}{8192}a_2^3-\frac{45}{128}a_2a_3+\frac{35}{256}a_4+\left(\frac{6093}{4096}a_2^3-\frac{1635}{512}a_2a_3+\frac{455}{256}a_4\right)\left(\frac{1}{s^2}\right)\right](s\epsilon)^3$$

$$-\left[\frac{197725}{524288}a_2^4-\frac{13981}{16384}a_2^2a_3+\frac{209}{1024}a_3^2+\frac{1617}{4096}a_2a_4-\frac{63}{512}a_5\right.$$

$$+\left(\frac{698497}{131072}a_2^4-\frac{29123}{2048}a_2^2a_3+\frac{8047}{2048}a_3^2+\frac{34923}{4096}a_2a_4-\frac{1827}{512}a_5\right)\left(\frac{1}{s^2}\right)$$

$$\left.+\left(\frac{709267}{131072}a_2^4-\frac{72751}{4096}a_2^2a_3+\frac{847}{128}a_3^2+\frac{27279}{2048}a_2a_4-\frac{63}{8}a_5\right)\left(\frac{1}{s^4}\right)\right](s\epsilon)^4$$

$$+\begin{cases}O(\epsilon^5), & \text{for TE-modes}\\ \left(1-\frac{3}{2}a_2\right)\left(\frac{1}{s^2}\right)(s\epsilon)^2+\left(\frac{21}{16}-\frac{21}{8}a_2-\frac{45}{32}a_2^2+\frac{45}{16}a_3\right)\left(\frac{1}{s^2}\right)(s\epsilon)^3 & \\ +\left[\left(\frac{25}{16}-\frac{981}{256}a_2-\frac{639}{512}a_2^2+\frac{135}{32}a_3-\frac{2541}{1024}a_2^3+\frac{1575}{256}a_2a_3-\frac{35}{8}a_4\right)\left(\frac{1}{s^2}\right)\right. & \\ \left.+\left(\frac{19}{4}-\frac{1827}{128}a_2-\frac{441}{256}a_2^2+\frac{135}{8}a_3-\frac{3003}{512}a_2^3+\frac{2205}{128}a_2a_3-\frac{35}{2}a_4\right)\left(\frac{1}{s^4}\right)\right](s\epsilon)^4+O(\epsilon^5), & \text{for TM-modes}\end{cases}$$

$$d_1=-\frac{1}{8}a_2-\left(\frac{17}{192}a_2^2-\frac{5}{48}a_3\right)s\epsilon$$

$$-\left[\frac{509}{4096}a_2^3-\frac{55}{256}a_2a_3+\frac{35}{384}a_4+\left(\frac{935}{2048}a_2^3-\frac{295}{256}a_2a_3+\frac{105}{128}a_4\right)\left(\frac{1}{s^2}\right)\right](s\epsilon)^2$$

$$-\left[\frac{3653}{16384}a_2^4-\frac{4217}{8192}a_2^2a_3+\frac{131}{1024}a_3^2+\frac{63}{256}a_2a_4-\frac{21}{256}a_5\right.$$

$$\left.+\left(\frac{76121}{32768}a_2^4-\frac{27275}{4096}a_2^2a_3+\frac{1975}{1024}a_3^2+\frac{2275}{512}a_2a_4-\frac{525}{256}a_5\right)\left(\frac{1}{s^2}\right)\right](s\epsilon)^3$$

$$+\begin{cases}O(\epsilon^4), & \text{for TE-modes}\\ \left(\frac{7}{8}-2a_2-\frac{9}{16}a_2^2+\frac{15}{8}a_3\right)\left(\frac{1}{s^2}\right)(s\epsilon)^2 & \\ +\left(\frac{25}{24}-\frac{87}{32}a_2-\frac{133}{192}a_2^2+\frac{145}{48}a_3-\frac{153}{128}a_2^3+\frac{55}{16}a_2a_3-\frac{35}{12}a_4\right)\left(\frac{1}{s^2}\right)(s\epsilon)^3+O(\epsilon^4), & \text{for TM-modes}\end{cases}$$

$$d_2=-\left(\frac{11}{384}a_2^2-\frac{1}{12}a_3\right)-\left(\frac{103}{2048}a_2^3-\frac{15}{128}a_2a_3+\frac{7}{96}a_4\right)s\epsilon$$

$$-\left[\frac{19579}{196608}a_2^4-\frac{16537}{61440}a_2^2a_3+\frac{661}{7680}a_3^2+\frac{763}{5120}a_2a_4-\frac{21}{320}a_5\right.$$

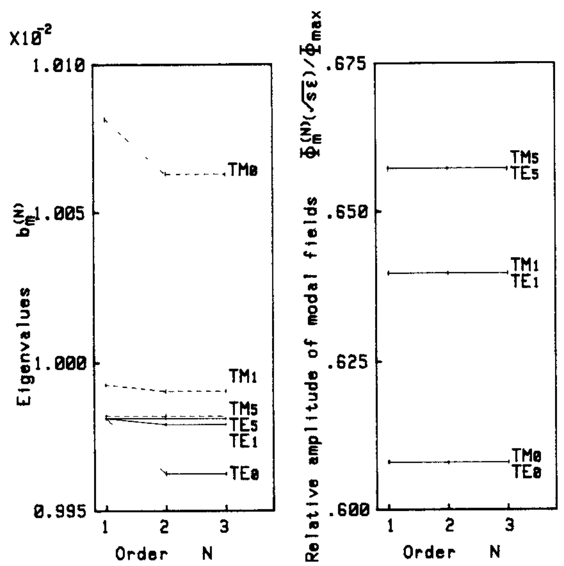
$$\left.+\left(\frac{52667}{98304}a_2^4-\frac{57221}{30720}a_2^2a_3+\frac{361}{480}a_3^2+\frac{7833}{5120}a_2a_4-\frac{21}{20}a_5\right)\left(\frac{1}{s^2}\right)\right](s\epsilon)^2$$

$$+\begin{cases}O(\epsilon^3), & \text{for TE-modes}\\ \left(\frac{5}{6}-\frac{177}{64}a_2+\frac{235}{384}a_2^2+\frac{31}{12}a_3-\frac{77}{256}a_2^3+\frac{79}{64}a_2a_3-\frac{7}{3}a_4\right)\left(\frac{1}{s^2}\right)(s\epsilon)^2+O(\epsilon^3), & \text{for TM-modes}\end{cases}$$

$$d_3=-\left(\frac{35}{3072}a_2^3-\frac{1}{24}a_2a_3+\frac{1}{16}a_4\right)$$

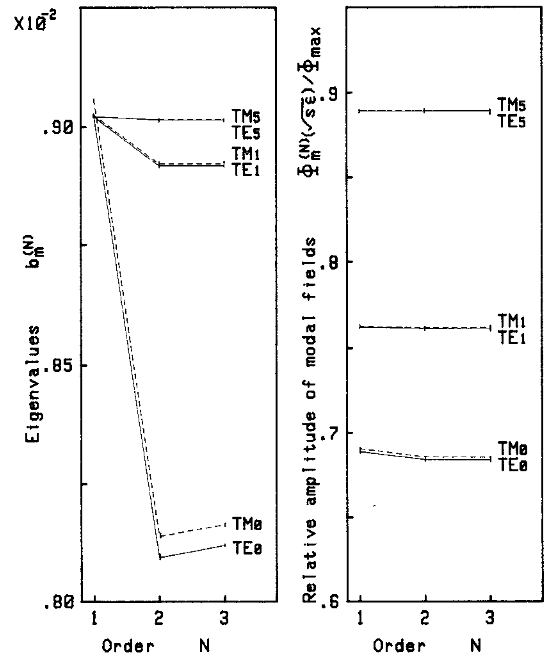
$$-\left(\frac{1183}{36864}a_2^4-\frac{10049}{92160}a_2^2a_3+\frac{241}{5760}a_3^2+\frac{169}{1920}a_2a_4-\frac{9}{160}a_5\right)s\epsilon+O(\epsilon^2)$$

$$d_4=-\left(\frac{1693}{294912}a_2^4-\frac{607}{23040}a_2^2a_3+\frac{23}{1440}a_3^2+\frac{21}{640}a_2a_4-\frac{1}{20}a_5\right)+O(\epsilon)$$



(a) (b)

Fig. 1. Convergence check of near-parabolic profile case when $s\epsilon = 0.01$. (a) Eigenvalues $b_m^{(N)}$. (b) Relative amplitude of modal fields $\Phi_m^{(N)}(\sqrt{s\epsilon})/\Phi_{\max}$.



(a) (b)

Fig. 2. Convergence check of quasi-Gaussian profile case when $s\epsilon = 0.01$. (a) Eigenvalues $b_m^{(N)}$. (b) Relative amplitude of modal fields $\Phi_m^{(N)}(\sqrt{s\epsilon})/\Phi_{\max}$.

where $s = 2m+1$. On the other hand, in the process of the calculation of (15), we obtain the WKB solution

$$\begin{aligned} b_m^{(1)} &= s\epsilon - \frac{3}{8}a_2(s\epsilon)^2 - \left(\frac{17}{64}a_2^2 - \frac{5}{16}a_3\right)(s\epsilon)^3 - \left(\frac{375}{1024}a_2^3 - \frac{165}{256}a_2a_3 + \frac{35}{128}a_4\right)(s\epsilon)^4 \\ &\quad - \left(\frac{10689}{16384}a_2^4 - \frac{3129}{2048}a_2^2a_3 + \frac{393}{1024}a_3^2 + \frac{189}{256}a_2a_4 - \frac{63}{256}a_5\right)(s\epsilon)^5 \\ &\quad + \begin{cases} O(\epsilon^6), & \text{for TE-modes} \\ \left(\frac{1}{s^2}\right)(s\epsilon)^2 + (2-3a_2)\left(\frac{1}{s^2}\right)(s\epsilon)^3 + \left(\frac{21}{8} - \frac{45}{8}a_2 - \frac{9}{4}a_2^2 + \frac{45}{8}a_3\right)\left(\frac{1}{s^2}\right)(s\epsilon)^4 \\ \quad + \left[\left(\frac{25}{8} - \frac{261}{32}a_2 - \frac{29}{16}a_2^2 + \frac{35}{4}a_3 - \frac{255}{64}a_2^3 + \frac{345}{32}a_2a_3 - \frac{35}{4}a_4\right)\left(\frac{1}{s^2}\right)\right. \\ \quad \left. + \left(-2 + 6a_2 - \frac{9}{2}a_2^2\right)\left(\frac{1}{s^4}\right)\right](s\epsilon)^5 + O(\epsilon^6), & \text{for TM-modes.} \end{cases} \end{aligned} \tag{17}$$

Comparison of (15) and (17) indicates that in the Nth-order approximate solution, the terms $1/s^i (2 \leqslant i \leqslant 2N-2)$ are added to the WKB solution. The WKB solution is very accurate for higher order modes, but not so accurate for the lower order modes, as expected. As a matter of course, the second-order approximate eigenvalues $b_m^{(2)}$ are in agreement with the previous ones [7], [8]. Moreover, we calculate $b_m^{(3)}$ by using the formula given in [6] and obtain (15) again.

V. Convergence Check

Here we examine the convergence of the approximate eigenvalues and modal fields of the guided modes of waveguide with a profile:

$$h(x) = K\left(1 - \exp\left(-(2\Delta/K)(x/D)^2\right)\right), \qquad 2\Delta \leqslant K \leqslant 1 \tag{18}$$

where Δ is the relative refractive-index difference between the guiding layer and the cladding, and D is the core width. We can realize the near-parabolic profile with $K = 1$ and the Gaussian profile with $K = 2\Delta$. We approximately trace the profile (18) by setting $a_M = (1/M!)(1/K^{M-1})$ and $g = \sqrt{2\Delta}/D$ in (1). There exist the guided modes for $b_m < 2\Delta$. On the other hand, from (15) we have $s\epsilon = 0(b_m)$. Then the range of $s\epsilon$ to be considered is $0 < s\epsilon = 0(2\Delta)$.

As a numerical check, we compute the third-order approximate eigenvalues and modal fields with $\Delta = 0.01$ in two cases; the near-parabolic profile and the quasi-Gaussian profile considering the terms up to a_5. Fig. 1 gives $b_m^{(N)}$ and $\Phi_m^{(N)}(\sqrt{s\epsilon})/\Phi_{\max}(m = 0,1,5.\ N = 1,2,3)$ for TE modes and TM modes in the case of the near-parabolic profile when $s\epsilon = 0.01$, where $\Phi_{\max}$ is the maximum amplitude of $\Phi_m^{(N)}(w_1)$. This figure shows that the calculated values of all the guided modes are converging. The third-order approximate solution is very accurate. The WKB solution for eigenvalues of lower order modes is significantly corrected. For modal fields, the WKB solution coincides with the refined solution with six significant figures. Fig. 2 gives $b_m^{(N)}$ and $\Phi_m^{(N)}(\sqrt{s\epsilon})/\Phi_{\max}$ for TE modes and TM modes in the case of the quasi-Gaussian profile when $s\epsilon = 0.01$. This figure indicates that the WKB solution for higher order modes is more accurate than that for lower order modes. The correction to the WKB solution

TABLE I
EIGENVALUES b_m IN THE CASES OF THE NEAR-PARABOLIC PROFILE (UPPER) AND THE QUASI-GAUSSIAN PROFILE (LOWER)

mode	$s\epsilon=.003$	$s\epsilon=.01$	$s\epsilon=.02$
TE_0	$.2996624579 \times 10^{-2}$	$.99624845 \times 10^{-2}$	$.1984988 \times 10^{-1}$
	$.2830 \times 10^{-2}$	$.8 \times 10^{-2}$	—
TM_0	$.3005638110 \times 10^{-2}$	$1.00629883 \times 10^{-2}$	$.2025394 \times 10^{-1}$
	$.2838 \times 10^{-2}$	$.8 \times 10^{-2}$	—
TE_1	$.2998124609 \times 10^{-2}$	$.9979152165 \times 10^{-2}$	$.199165504 \times 10^{-1}$
	$.290524 \times 10^{-2}$	$.892 \times 10^{-2}$	$.16 \times 10^{-1}$
TM_1	$.2999126111 \times 10^{-2}$	$.9990319066 \times 10^{-2}$	$.199624430 \times 10^{-1}$
	$.290603 \times 10^{-2}$	$.892 \times 10^{-2}$	$.16 \times 10^{-1}$
TE_5	$.2998298166 \times 10^{-2}$	$.9981080673 \times 10^{-2}$	$.1992426494 \times 10^{-1}$
	$.29139235 \times 10^{-2}$	$.90139 \times 10^{-2}$	$.158 \times 10^{-1}$
TM_5	$.2998372658 \times 10^{-2}$	$.9981911267 \times 10^{-2}$	$.1992760403 \times 10^{-1}$
	$.29139821 \times 10^{-2}$	$.90142 \times 10^{-2}$	$.158 \times 10^{-1}$

is significant for lower order modes. This figure also shows that a higher order asymptotic solution is required for analyzing lower order modes. Lastly, we examine the significant figures of the calculated eigenvalues. Table I shows that the significant figures in the case of the near-parabolic profile are more than seven.

VI. CONCLUSION

We present the algorithm for calculating the propagation constants and the modal fields of the guided modes in even polynomial refractive-index media. The algorithm presented here leads to a highly accurate solution in the sense of the asymptotic expansion. The third-order approximate solutions of the propagation constants and the field distributions are derived in analytic form. The convergence and the accuracy of solution for the guided modes of waveguides with the near-parabolic profile and the quasi-Gaussian profile are examined numerically. It is found that the third-order asymptotic solution is accurate for the guided modes of the near-parabolic profile waveguides and for higher order modes in the case of the quasi-Gaussian profile. In order to evaluate lower order modes in complex profiles such as the Gaussian profile, we need to calculate a higher order asymptotic solution.

The analysis of the guided modes of cladded inhomogeneous slab waveguides with uniform outer layers is carried out in a companion paper [12].

APPENDIX
ESTIMATE OF R_N

In the case of the even polynomial refractive-index profile, from (9) or (13), we have $b_m = O(\epsilon)$. So we get $\xi_1^2 = O(b_m) = O(\epsilon)$ and $w_1 = O(\xi_1) = O(\epsilon^{1/2})$. From the above fact and (5), we get a family of relations

$$\begin{aligned} ew_1 &= w_2 + f_1 w_2^3 + g_1 w_2^5 + \cdots \\ w_2 &= w_3 + f_2 w_3^3 + g_2 w_3^5 + \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ w_p &= w_{p+1} + f_p w_{p+1}^3 + g_p w_{p+1}^5 + \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ w_N &= w_{N+1} + f_N w_{N+1}^3 + g_N w_{N+1}^5 + \cdots \end{aligned} \tag{A1}$$

where $e = O(\epsilon^{-1/2})$, $f_p = O(\epsilon^{2p-1})$, and $g_p = O(\epsilon^{2p})$ $(p = 1,2,\cdots,N)$.

It is noted that e, f_p, and g_p are expressed in the power series of ϵ. Substituting (A1) into (5d), we have the estimate of R_N such that

$$R_N = O(\epsilon^{2N-1}) \tag{A2}$$

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